Gluing methods in almost-Kähler geometry

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Inspiration: Uniformization theorem for Riemann surfaces

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- Framework: (almost)-Kähler manifolds
- 'Privileged' metrics = constant scalar curvature metrics in a fixed Kähler class.

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- Obstructions related to the existence of holomorphic vector fields.
- Tian-Yau-Donaldson conjecture: hamiltonian action on the space of (almost-)complex structures compatible with the Kähler form. The moment map is then the (normalized) scalar curvature $s(\omega) - \bar{s}$.
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Gluing methods are a way of obtaining explicit examples.
Examples: Works of Arezzo and Pacard, Szekelyhidi.

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(M, J_M, ω_M) a compact Kähler orbifold, with a single isolated singularity p modelled on $\mathbb{C}^m/\Gamma$, where $\Gamma \subset U(m)$ only fixes 0.

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From the Kähler property, we have holomorphic coordinates around p:

$$\underline{z} : p \ni U \to U' \subset \mathbb{C}^m/\Gamma$$

such that

$$\omega_M = \omega_{eucl} + O(|\underline{z}|^2).$$

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We assume that (X, J_X, ω_X) has zero scalar curvature.

The 'connected sum'

Using these coordinates, we identify a small 'ring' around $p \in M$ to a large region in X via a homothety.

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Thus we obtain a *complex* smooth manifold M_ε.

We endow it with a Kähler form by joining Kähler potentials for ω_M and ω_X by cut-off functions.

Then, the problem is to find $f \in \mathcal{C}^\infty(M_\varepsilon)$ such that

$$s(\omega_\varepsilon + dd^c f) = \lambda.$$

Definition

An *almost-Kähler* manifold is a symplectic manifold (V, ω) endowed with a compatible almost-complex structure J.

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- Lejmi, Keller and Lejmi: study of Calabi's functional, Futaki's invariant
- Weinkove: Calabi-Yau equation.

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Question: Could we use gluing methods to obtain privileged metrics on almost-Kähler manifolds ?

We use more general ALE models (X, J_X, ω_X):

$$\underline{u} : X \setminus K \to \mathbb{C}^m/\Gamma$$

diffeomorphism outside a compact set, such that

$$\omega_X = \omega_{eucl} + O(|\underline{u}|^{2-2m})$$
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Such non-trivial resolutions only exist in complex dimension 2 (Hein, Radeasconu, Suvaina).

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  We explicitly obtain the ALE metric by solving $\mathrm{Ric}(dd^c\varphi) = 0$, with $\varphi = f(|z|^2)$, on *smoothings*

$$\{z_1^2 + z_2^2 + z_3^2 = \varepsilon\} \subset \mathbb{C}^3$$

  of

$$\mathbb{C}^2/\mathbb{Z}_2 \simeq \{z_1^2 + z_2^2 + z_3^2 = 0\}.$$

  The Riemannian metric g_X thus derived is Eguchi-Hanson's. The complex structure inherited from $\mathbb{C}^3$ verifies

$$J_X - J_{eucl} = O(|u|^{-4}).$$

Generalized connected sum.

We work in Darboux charts.

- On the orbifold: an equivariant version of Darboux's theorem near p gives
$$\underline{x} : p \ni U \to U' \subset \mathbb{C}^2/\mathbb{Z}_2$$
such that ω_M coincides with ω_{eucl}. Moreover
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- On the ALE surface: Performing an explicit change of variables, we get
$$\underline{u} : X \setminus K \to (\mathbb{C}^2 \setminus B(0,R))/\mathbb{Z}_2$$
such that ω_X coincides with ω_0. Moreover, we recover
$$J_X - J_{eucl} = O(|\underline{u}|^{-4}).$$

Using these charts to perform the connected sum construction gives us a family of *symplectic* manifolds $(M_\varepsilon, \omega_\varepsilon)$.

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1. The $(M_\varepsilon, \omega_\varepsilon)$ are all symplectically equivalent to the same $(\tilde{M}, \tilde{\omega})$;

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2. The zero section of $X \simeq T^*S^2$ provides a Lagrangian sphere S in $(\tilde{M}, \tilde{\omega})$.

We want to

- endow M_ε with an almost complex structure compatible with ω_ε.
- perturb this structure into a 'canonical' one.

Almost complex structures on $(M_\varepsilon, \omega_\varepsilon)$.

We use the following description of $\mathcal{AC}_\omega$ on a symplectic manifold (V, ω).

Theorem

Let (V, ω) be a symplectic manifold. We set

$$\mathrm{End}(TV, \omega) = \{a \in \mathrm{End}(TV), \, \omega(aX, Y) + \omega(X, aY) = 0\},$$

the Lie algebra of automorphisms of TV that preserve ω.
Then, if $J_1, J_2 \in \mathcal{AC}_\omega$, there exists $a \in \mathcal{C}^\infty(\mathrm{End}(TV, \omega))$ such that

$$J_2 = \exp(a) J_1 \exp(-a).$$

Using this description and suitable cut-off functions, we get

- an a.c.s. J_{r_ε} on M, compatible with ω_M, such that

$$J_{r_\varepsilon} = \begin{cases} J_0 \text{ on } \{|\underline{x}| \leq 2r_\varepsilon\} \\ J_M \text{ on } \{|\underline{x}| \geq 4r_\varepsilon\} \end{cases}$$

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Remarks:

1. J_ε is not integrable, but N_{J_ε} is supported in the gluing region $\{r_\varepsilon \leq |\underline{x}| \leq 4r_\varepsilon\}$, and controlled in suitable norms by a positive power of ε.
2. The Lagrangian sphere S_ε is minimal for g_ε.

Perturbation of the approximate solution

Problem: For $f \in \mathcal{C}^\infty(M_\varepsilon)$, the form

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is not J_ε-invariant.

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and we set

$$J_f := \exp(-a_f) J_\varepsilon \exp(a_f).$$

The equation

We want to solve

$$s^{\nabla}(J_f) = s_M + \lambda, \tag{1}$$

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The equation (1) is a fourth order PDE on f.

Strategy.

We imitate the proof of the Inverse Function Theorem. We linearise:

$$L_\varepsilon(f) := \frac{d}{dt}_{|t=0} s^\nabla(J_{tf}),$$

thus

$$s^\nabla(J_f) = s^\nabla(J_\varepsilon) + L_\varepsilon(f) + Q_\varepsilon(f).$$

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The linearised operator is given by

$$\begin{aligned} L_\varepsilon f &= -\Delta^2_{g_\varepsilon} f + 2\delta_{g_\varepsilon} \mathrm{Ric}_{g_\varepsilon}(\mathrm{grad}_{g_\varepsilon} f, \cdot) + E_\varepsilon f \\ &= \mathbb{L}_{M_\varepsilon} f + E_\varepsilon f, \end{aligned}$$

where $\mathbb{L}$ is the *Lichnerowicz operator* on M_ε, and the error term E_ε is small in suitable norms, with coefficients comparable to the Nijenhuis tensor N_{J_ε}.

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1. Build a right inverse $\tilde{L}_{\varepsilon}(f, \lambda) = L_{\varepsilon} f + \lambda$ in suitable Banach spaces,
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3. Control the non-linear term Q_{ε}.

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These operators are defined on *noncompact* manifolds: in terms of regularity property, they do not behave well in usual Hölder spaces $\mathcal{C}^{k,\alpha}(M^*)$, $\mathcal{C}^{k,\alpha}(X)$ (ex: Schauder estimates are lost).

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Definition

- **On** M^***:** $\phi \in \mathcal{C}^{k,\alpha}_{\delta}(M^*)$ if $\phi \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(M^*)$ and ϕ behaves 'at worst' like $|\underline{x}|^{\delta}$ near the puncture p.

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- **On X:** $\psi \in \mathcal{C}^{k,\alpha}_{\delta}(X)$ if $\psi \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(X)$ and ψ behaves 'at worst' like $|\underline{u}|^{\delta}$ at infinity.

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- **On** X**:** $\psi \in \mathcal{C}^{k,\alpha}_{\delta}(X)$ if $\psi \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(X)$ and ψ behaves 'at worst' like $|\underline{u}|^{\delta}$ at infinity.
- **On** M_ε**:** We decompose $f \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(M_\varepsilon)$ into $f = \gamma_1 f + \gamma_2 f$, où $\gamma_1 f \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(M^*)$ and $\gamma_2 f \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(X)$. Then we set

$$\|f\|_{\mathcal{C}^{k,\alpha}_{\delta}(M_\varepsilon)} = \|\gamma_1 f\|_{\mathcal{C}^{k,\alpha}_{\delta}(M^*)} + \varepsilon^{-\delta}\|\gamma_2 f\|_{\mathcal{C}^{k,\alpha}_{\delta}(X)}.$$

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Proposition

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- Let $\xi \in \mathcal{C}^\infty(M)$ supported in $B(p, 2r_0)$ and equal to 1 in $B(p, r_0)$. Then

$$\tilde{L}_{M^*} : (\mathcal{C}^{4,\alpha}_\delta(M^*) \oplus \mathrm{Vect}(\xi)) \times \mathbb{R} \to \mathcal{C}^{0,\alpha}_{\delta-4}(M^*)$$

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  admits a right inverse G_1;
- $\mathbb{L}_X : \mathcal{C}^{4,\alpha}_\delta(X) \to \mathcal{C}^{0,\alpha}_{\delta-4}(X)$ admits a right inverse G_2.

Right inverse for $\tilde{L}_\varepsilon$

From there we get:

Theorem

For $0 < \delta < 1$, for ε small enough, the operator

$$\tilde{L}_\varepsilon : \mathcal{C}^{4,\alpha}_{\delta}(M_\varepsilon) \times \mathbb{R} \to \mathcal{C}^{0,\alpha}_{\delta-4}(M_\varepsilon)$$

admits a right inverse G_ε, such that $\|G_\varepsilon\| \leq \varepsilon^{-\delta\beta}$, with $0 < \beta < 1$.

Outline of proof: We glue together G_1 and G_2 into an 'approximate right inverse': for $f \in \mathcal{C}^{0,\alpha}_{\delta-4}(M_\varepsilon)$, se set

$$\tilde{G}_\varepsilon(f) = \zeta_1 G_1(\gamma_1 f) + \zeta_2 G_2(\gamma_2 f).$$

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Then we show that:

$$\|\tilde{L}_\varepsilon \circ \tilde{G}_\varepsilon - I\| \xrightarrow{\varepsilon \to 0} 0.$$

Thus, $G_\varepsilon := \tilde{G}_\varepsilon \circ (\tilde{L}_\varepsilon \circ \tilde{G}_\varepsilon)^{-1}$ is a genuine right inverse for $\tilde{L}_\varepsilon$.

Definition. Euler-Lagrange equation.

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Let (V, ω, J, g) be an almost-Kähler manifold. A Lagrangian submanifold L of V is *Hamiltonian-stationary* if

$$\frac{d}{ds}_{|s=0} Vol_g(\exp(sX_F)(L)) = 0$$

for any $F \in \mathcal{C}^\infty(L)$.

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Let H be the mean curvature vector field of L. We define the *Maslov-form* $\alpha := H \lrcorner \omega$.
Then the Euler-Lagrange equation associated to the variational problem is

$$\delta\alpha = 0.$$

Construction of Hamiltonian-stationary spheres.

In our setting, we essentially obtained a sympectic manifold $(\tilde{M}, \tilde{\omega})$ endowed with

- a Lagrangian sphere S;
- a family of metrics with constant Hermitian scalar curvature $(\tilde{J}_\varepsilon, \tilde{g}_\varepsilon)$.

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Question: For ε small enough, can we find a function F_ε such that $\exp(X_{F_\varepsilon})(S)$ be Hamiltonian-stationary for $(\tilde{g}_\varepsilon, \tilde{J}_\varepsilon)$?

Answer: Yes ! The idea is to study the operator

$$\begin{aligned} B : \mathcal{C}^{2,\alpha}(\mathcal{AC}_{\tilde{\omega}}) \times \mathcal{C}^{4,\alpha}(S) &\to \mathcal{C}^{0,\alpha}(S) \\ (J, F) &\mapsto \delta_{J,F}\alpha_{J,F} \end{aligned}$$

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We have $B(\tilde{J}_0, 0) = 0$.

On the other hand, the linearisation of B at $(\tilde{J}_0, 0)$ with respect to the second variable is $\Delta^2_{\tilde{g}_0}$ (Oh's formula).

This allows us to use the Implicit Function Theorem.

Conclusion and perspectives

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Perpectives

- Other types of singularities;
- Higher dimensions, for instance smoothings of double points;
- Can the blow-up construction be made in a way to preserve the constant curvature condition ?

Thank you for your attention !